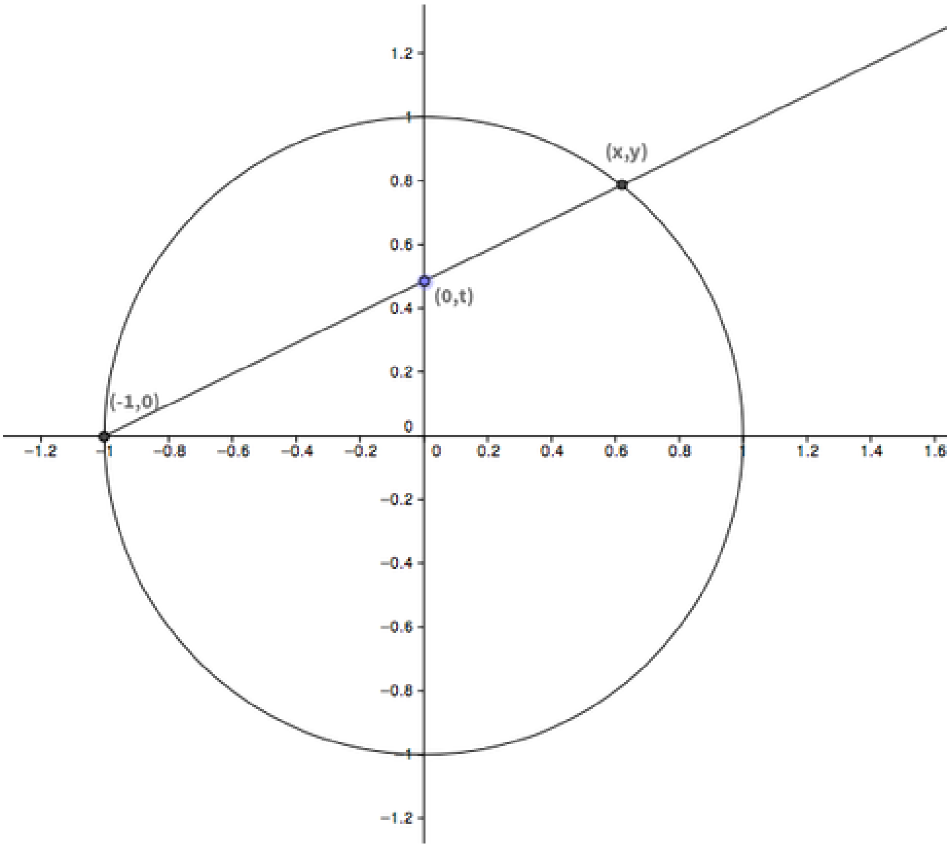
Rational parameterization of the circle



1. Pick a rational number t. Positive, negative, small, large - doesn't matter.

2. Place a point at $(0, t)$.

3. Fire a laser beam from $(-1, 0)$through your point $(0, t)$.

4. The laser beam meets the unit circle (the circle of radius 1 around the origin) at some point (x, y).

5. The numbers $x$and $y$are always rational. Guaranteed.

6. Since $(x, y)$is on the unit circle, you're also guaranteed that $x^2 + y^2 = 1$.

7. So, find those rational numbers $x$and y,write down the relation $x^2 + y^2 = 1$, multiply by the denominators, and there you have your Pythagorean triplet.

Example: let's pick $t = \frac{1}{2}$. The laser beam is then the line $y = x/2 + 1/2$(you can easily check that this line passes through $(-1, 0)$and $(0, 1/2)$). Solving a little equation you find that this line meets the circle again at $(\frac{3}{5}, \frac{4}{5})$. Therefore $(3/5)^2 + (4/5)^2 = 1$which is the same thing as $3^2 + 4^2 = 5^2$.

You can carry the full calculation through for arbitrary t, and the result is

$$(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

This is the not-sufficiently-famous rational parametrization of the circle, which can be used to explain the parametric formula for pythagorean triplets as well as the importance of the substitution $t = \tan(\theta/2)$in resolving trigonometric integrals, and other things besides.

The general fact that underlies all this is that a rational conic with a rational point always has a rational parametrization. This is a basic fact in algebraic geometry.